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Point Processes and the Position Distribution of Infinite Boson Systems

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It is shown that to each locally normal state of a boson system one can associate a point process that can be interpreted as the position distribution of the state. The point process contains all information one can get by position measurements and is determined by the latter. On the other hand, to each so-called Σ^c-point process Q we relate a locally normal state with position distribution Q.

KEY WORDS: Point processes; Poisson processes; Campbell measure; compound Campbell measure; boson systems; locally normal states; Glauber states; reduced density matrix.

1. INTRODUCTION

The state of a finite system of bosons in $\mathbb{R}^d$ is described by a normalized positive trace-class operator on the symmetric Fock space over $\mathbb{R}^d$. In order to characterize infinite systems one may use positive normalized linear functionals ω on a suitably chosen C^*-algebra $\mathscr{A}$ of bounded operators on the Fock space.

Let f be a bounded measurable function on the space of all locally finite point configurations in $\mathbb{R}^d$, and denote by O_f the corresponding bounded operator of multiplication on the Fock space. In Section 3 (Theorem 3.2) we will prove that there exists exactly one point process Q on $\mathbb{R}^d$ such that

$$\omega(O_f) = \int f \cdot dQ, \qquad O_f \in \mathscr{A}$$

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Each position measurement can be expressed by an operator O_f. The $\omega(O_f)$ is interpreted as the expectation of the position measurement O_f in the state ω. A point process is the probability distribution of a locally finite random point system Φ in $\mathbb{R}^d$. Thus, $\int f\,dQ$ is the expectation of the random variable $f(\Phi)$. For that reason Q may be interpreted as the distribution law of the position vector of the quantum mechanical particle system, and we will call Q the position distribution of ω.

Further, we will discuss general properties of such position distributions. In particular, we will show that the position distributions are locally so-called Σ^c-processes. On the other hand, to each Σ^c-point process (finite Σ^c-point process) Q we can relate a locally normal state (normal state) with position distribution Q (Theorem 3.3). Let us remark that the proof of Theorem 3.3 is based on an explicit "construction" of a state with position distribution Q. So the proof indicates a way to find many nontrivial examples of states of infinite boson systems.

A state of a quantum system never will be characterized solely by its position distribution (unless the system is not a classical one). Besides the position distribution, we still need one special function (which we called the conditional reduced density matrix) to determine the state completely. The idea is to consider measurements that can be divided into two parts—an application of a local observable to a finite subsystem and a position measurement to the possibly infinite "rest" configuration. These investigations can be found in Refs. 4 and 5. In Ref. 3 we considered pure states of boson systems.

In the present paper we reduce all our considerations to locally normal states of boson systems without spin. We further assume that the local algebras consist of all bounded linear operators on the Fock spaces over the bounded regions of the phase space $\mathbb{R}^d$. Though certain generalizations can be made in an easy way, we will not touch in this paper the problem of passing over to fermion systems, to systems with spin, or to other reasonable phase spaces.

2. BASIC NOTIONS AND NOTATIONS

2.1. Counting Measures and Point Processes

Let $[\mathbb{R}^d, \mathfrak{R}^d]$, $d \geqslant 1$ denote the d-dimensional Euclidean space equipped with the σ-algebra of Borel subsets, $\mathfrak{B}$ the ring of bounded sets in $\mathfrak{R}^d$, δ_x the Dirac measure in $x \in \mathbb{R}^d$, and $\mathbf{N}$ the set of nonnegative integers. Further, let M be the set of all locally finite integer-valued measures on $[\mathbb{R}^d, \mathfrak{R}^d]$, i.e.,

$$M = \{\varphi : \varphi \text{ is a measure on } [\mathbb{R}^d, \mathfrak{R}^d],\ \varphi(\Lambda) \in \mathbf{N} \text{ for all } \Lambda \in \mathfrak{B}\}$$

The elements of M, which are called counting measures, can be interpreted as locally finite point configurations in $\mathbb{R}^d$. Indeed, a measure φ on $[\mathbb{R}^d, \mathfrak{R}^d]$ belongs to M if and only if φ can be written in the form $\varphi = \sum_{j\in J} \delta_{xj}$ with J a finite or countable index set, $x_j \in \mathbb{R}^d$ for all $j \in J$, and $(x_j)_{j\in J}$ having no accumulation points [each $\Lambda \in \mathfrak{B}$ contains only finitely many points from $(x_j)_{j\in J}$]. Thus, a counting measure $\varphi = \sum_{j\in J} \delta_{xj}$ represents a locally finite point system with position vector $(x_j)_{j\in J}$.

For arbitrary $\Lambda \in \mathbb{R}^d$ denote by ${}_{\Lambda}\mathfrak{M}$ the smallest σ-algebra of subsets of M that makes the mappings $\xi_{\Lambda'}$: $M \to \mathbf{N}$ defined by

$$\xi_{\Lambda'}(\varphi) = \varphi(\Lambda'), \qquad \varphi \in M \tag{2.1}$$

measurable for each $\Lambda' \in \mathfrak{B} \cap \Lambda$. In particular, we set ${}_{\mathbb{R}^d}\mathfrak{M} = \mathfrak{M}$. We still introduce some important subsets of M. By M^s we denote the set of simple counting measures (having no multiple points), i.e., $M^s = \{\varphi \in M$: $\varphi(\{x\}) \leqslant 1$ for all $x \in \mathbb{R}^d\}$. The set M^f is the set of finite counting measures, i.e., $M^f = \{\varphi \in M$: $\varphi(\mathbb{R}^d) < \infty\}$. For arbitrary $\Lambda \in \mathfrak{R}^d$ we denote by M_Λ the set of counting measures concentrated on Λ, i.e., $M_\Lambda = \{\varphi \in M$: $\varphi(\Lambda^c) = 0\}$, where $\Lambda^c = \mathbb{R}^d \backslash \Lambda$, and by v_Λ the restriction from M onto M_Λ, i.e. v_Λ: $M \to M_\Lambda$ is defined by

$$(v_\Lambda \varphi)(\cdot) \underset{\text{def}}{=} \varphi_\Lambda(\cdot) = \varphi(\cdot \cap \Lambda), \qquad \varphi \in M \tag{2.2}$$

Observe that the sets introduced above belong to $\mathfrak{M}$ and that v_Λ is measurable for each $\Lambda \in \mathfrak{R}^d$.

Further, we set $\mathfrak{M}_\Lambda = \mathfrak{M} \cap M_\Lambda$, $\Lambda \in \mathfrak{R}^d$. In the sequel we always have to distinguish carefully between $\mathfrak{M}_\Lambda$ and ${}_\Lambda\mathfrak{M}$. Counting measures from a set $Y \in \mathfrak{M}_\Lambda$ have no mass points outside Λ, while the sets from ${}_\Lambda\mathfrak{M}$ are determined by the behavior of their elements inside Λ, i.e., $\mathfrak{M}_\Lambda = \{v_\Lambda Y$: $Y \in \mathfrak{M}\}$, ${}_\Lambda\mathfrak{M} = \{v_\Lambda^{-1} Y$: $Y \in \mathfrak{M}_\Lambda\}$.

Definition 2.1. A point process is a probability measure on $[M, \mathfrak{M}]$. A point process Q is called simple if $Q(M^s) = 1$, and is said to be finite if $Q(M^f) = 1$.

An important notion in point process theory is the so-called reduced Campbell measure.

Definition 2.2. Let Q be a point process and n a positive integer. The nth-order reduced Campbell measure $C_Q^{(n)}$ is the measure on $[(\mathbb{R}^d)^n \times M, (\mathfrak{R}^d)^n \otimes \mathfrak{M}]$ characterized by

$$C_Q^{(n)}(\Lambda \times Y) = \int Q(d\varphi) \int_\Lambda \varphi(d\underline{x}^n)\, \chi_Y(\varphi - \delta_{\underline{x}^n}), \qquad \Lambda \in (\mathfrak{R}^d)^n, \quad Y \in \mathfrak{M} \tag{2.3}$$

Here $\underline{x}^n = (x_1,..., x_n)$, $x_j \in \mathbb{R}^d$ for $j \in \{1,..., n\}$, $\chi_{(\cdot)}$ denotes the indicator function, $\delta_{\underline{x}^n}$ is an abbreviation of $\sum_{j=1}^n \delta_{x_j}$, and $\varphi(d\underline{x}^n)$ is the so-called nth factorial measure of φ [on $(\mathfrak{R}^d)^n$] defined by

$$\varphi(d\underline{x}^n) = \varphi(dx_1)(\varphi - \delta_{x_1})(dx_2) \cdot \cdots \cdot \left(\varphi - \sum_{j=1}^{n-1} \delta_{xj}\right)(dx_n) \tag{2.4}$$

Observe that the measure $C_Q^{(n)}$ is σ-finite.

Finally, we introduce the notion of Σ^c-point processes, which play an important role in our further considerations.

Definition 2.3. A point process Q is said to be a Σ^c-point process if there exists a σ-finite measure S on $[M, \mathfrak{M}]$ (called a supporting measure of Q) such that

$$C_Q^{(1)} \ll l \times S$$

where $\ll$ denotes absolute continuity and l is the Lebesgue measure on $\mathbb{R}^d$.

For further and more detailed information about point process theory see Refs. 8 and 10. Characterizations of Σ^c-point processes can be found in Ref. 14, where this class of point processes was introduced.

2.2. The Symmetric Fock Space over Λ

The notion of the symmetric Fock space we want to introduce now is adapted to the language of counting measures.

For each $\Lambda \in \mathfrak{R}^d$ define a σ-finite measure F_Λ on $[M, \mathfrak{M}]$ by

$$F_\Lambda(Y) = \chi_Y(\mathrm{o}) + \sum_{n \geqslant 1} \frac{1}{n!} \int_{\Lambda^n} l^n(d\underline{x}^n)\, \chi_Y(\delta_{\underline{x}^n}), \qquad Y \in \mathfrak{M} \tag{2.5}$$

where o denotes the empty realization in M, i.e., $\mathrm{o}(\mathbb{R}^d) = 0$, and l^n is the Lebesgue measure on $(\mathbb{R}^d)^n$ $(l^1 = l)$.

We set $F_{\mathbb{R}^d} = F$. Observe that for $\Lambda \in \mathfrak{B}$ the measure F_Λ is a finite one $[F_\Lambda(M) = \exp\{l(\Lambda)\}]$, and that for all $\Lambda \in \mathfrak{R}^d$, F_Λ is concentrated on $M^s \cap M^f \cap M_\Lambda$. Thus, F_Λ is concentrated on all finite point configurations in Λ without multiple points.

The set of complex numbers we will denote by $\mathbb{C}$.

Definition 2.4. Let $\Lambda \in \mathfrak{R}^d$. The set

$$\mathscr{M}_\Lambda = \left\{ \Psi\colon M \to \mathbb{C},\ \Psi \text{ measurable, supp } \Psi \subseteq M_\Lambda \cap M^f, \right.$$

$$\left. \int F_\Lambda(d\varphi)\, |\Psi(\varphi)|^2 < \infty \right\}$$

equipped with the scalar product

$$(\Psi^1, \Psi^2)_\Lambda = \int F_\Lambda(d\varphi)\, \overline{\Psi^1(\varphi)}\, \Psi^2(\varphi)$$

we call the (symmetric) Fock space over Λ. The corresponding norm we denote by $\|\cdot\|_\Lambda$. By $\bar{\Psi}$ we denoted the complex conjugate of Ψ.

Further, for all $n \in \mathbf{N}$ and $\Lambda \in \mathfrak{R}^d$ we denote by $M_{n,\Lambda}$ the set of all n-point configurations in M_Λ, i.e.,

$$M_{n,\Lambda} = \{\varphi \in M_\Lambda : \varphi(\mathbb{R}^d) = \varphi(\Lambda) = n\}$$

Analogously, we define a set $\mathscr{M}_{n,\Lambda}$ by

$$\mathscr{M}_{n,\Lambda} = \{\Psi \in \mathscr{M}_\Lambda : \operatorname{supp} \Psi \subseteq M_{n,\Lambda}\}$$

For $\Lambda = \mathbb{R}^d$ we omit in all of the above notations the index Λ.

Remark 2.5. Usually one defines the symmetric Fock space over Λ as the direct sum $\bigoplus_{n \in \mathbf{N}} H_n$ of the symmetric subspaces H_n of $L_2(\Lambda^n)$ (with $H_0 = \mathbb{C}$) equipped with the inner product

$$((\Phi_n^1)_{n \in \mathbf{N}}, (\Phi_n^2)_{n \in \mathbf{N}}) = \overline{\Phi_0^1}\Phi_0^2 + \sum_{n \geq 1} \int_{\Lambda^n} l^n(d\underline{x}^n)\, \overline{\Phi_n^1(\underline{x}^n)}\, \Phi_n^2(\underline{x}^n)$$

(cf. Ref. 2, Chapter 1, §1.3; Ref. 1, Chapter 5.2.1).

Now we have the following orthogonal decomposition of $\mathscr{M}_\Lambda$:

$$\mathscr{M}_\Lambda = \bigoplus_{n \in \mathbf{N}} \mathscr{M}_{n,\Lambda}$$

It is easy to check that the operator $U = (U_n)_{n=0}^\infty$ given by

$$U_n(\Psi_n)(\underline{x}^n) = \frac{1}{n!}\, \Psi_n(\delta_{\underline{x}^n}), \qquad n \geq 1,\ \Psi_n \in \mathscr{M}_{n,\Lambda},\ \underline{x}^n \in \Lambda^n$$

$$U_0 \Psi_0 = \Psi_0, \qquad \Psi_0 \in \mathbb{C}$$

is a unitary operator from $\mathscr{M}_\Lambda$ onto $\bigoplus_{n \in \mathbf{N}} H_n$, i.e., the space $\bigoplus_{n \in \mathbf{N}} H_n$ is isomorphic to $\mathscr{M}_\Lambda$.

A definition of the symmetric Fock space similar to the construction of $\mathscr{M}_\Lambda$ was given, for instance, by Maasen.[9] But since he is dealing with the symmetric Fock space spanned up by a closed interval of $\mathbb{R}^1$, he can use the order of the real line and avoid using counting measures (cf. also Refs. 6 and 7).

Since we will deal in this paper only with boson systems, we call $\mathscr{M}_\Lambda$ for brevity the Fock space over Λ.

Denote by $\otimes$ the tensor product of Hilbert spaces, resp. of elements of them. The following connection between different $\mathscr{M}_\Lambda$ is true.

Proposition 2.6. Let Λ, $\Lambda' \in \mathfrak{R}^d$, $\Lambda \cap \Lambda' = \emptyset$. There exists a unique isomorphism $I_{\Lambda,\Lambda'}$ between $\mathscr{M}_\Lambda \otimes \mathscr{M}_{\Lambda'}$ and $\mathscr{M}_{\Lambda \cup \Lambda'}$ such that

$$I_{\Lambda,\Lambda'}(\Psi \otimes \Psi')(\varphi) = \Psi(\varphi_\Lambda)\, \Psi'(\varphi_{\Lambda'}) \qquad \Psi \in \mathscr{M}_\Lambda, \quad \Psi' \in \mathscr{M}_{\Lambda'}, \quad \varphi \in M_{\Lambda \cup \Lambda'} \tag{2.6}$$

[φ_Λ was defined in (2.2)].

In the sequel we will write $d\underline{x}^n$ instead of $l^n(d\underline{x}^n)$ and $d\underline{x}^0$ instead of $\delta_{\mathfrak{o}}$, i.e., for arbitrary $\Lambda \in \mathfrak{R}^d$ and g: $M \to \mathbb{C}$ we set

$$\int_{\Lambda^0} d\underline{x}^0 g(\delta_{\underline{x}^0}) = g(\mathfrak{o}) \tag{2.7}$$

First we are going to prove a lemma.

Lemma 2.7. Let Λ, $\Lambda' \in \mathfrak{R}^d$, $\Lambda \cap \Lambda' = \emptyset$, and h: $M \times M \to \mathbb{C}$ be a measurable, $F_\Lambda \times F_{\Lambda'}$-integrable function. Then

$$\int (F_\Lambda \times F_{\Lambda'})(d[\varphi_1, \varphi_2])\, h(\varphi_1, \varphi_2) = \int F_{\Lambda \cup \Lambda'}(d\varphi)\, h(\varphi_\Lambda, \varphi_{\Lambda'}) \tag{2.8}$$

Proof. Using the notation (2.7), we get

$$\begin{aligned}
&\int F_{\Lambda \cup \Lambda'}(d\varphi)\, h(\varphi_\Lambda, \varphi_{\Lambda'}) \\
&\quad = \sum_{n \in \mathbf{N}} \frac{1}{n!} \int_{(\Lambda \cup \Lambda')^n} d\underline{x}^n\, h((\delta_{\underline{x}^n})_\Lambda, (\delta_{\underline{x}^n})_{\Lambda'}) \\
&\quad = \sum_{n \in \mathbf{N}} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \int_{\Lambda^k} d\underline{x}^k \int_{(\Lambda')^{n-k}} d\underline{y}^{n-k}\, h(\delta_{\underline{x}^k}, \delta_{\underline{y}^{n-k}}) \\
&\quad = \sum_{k \in \mathbf{N}} \sum_{n=k}^{\infty} \frac{1}{k!\,(n-k)!} \int_{\Lambda^k} d\underline{x}^k \int_{(\Lambda')^{n-k}} d\underline{y}^{n-k}\, h(\delta_{\underline{x}^k}, \delta_{\underline{y}^{n-k}}) \\
&\quad = \sum_{n,k \in \mathbf{N}} \frac{1}{k!} \int_{\Lambda^k} d\underline{x}^k \frac{1}{n!} \int_{(\Lambda')^n} d\underline{y}^n\, h(\delta_{\underline{x}^k}, \delta_{\underline{y}^n}) \\
&\quad = \int (F_\Lambda \times F_{\Lambda'})(d[\varphi_1, \varphi_2])\, h(\varphi_1, \varphi_2)
\end{aligned} \tag{2.9}$$

Remark 2.8. From the chain (2.9) we conclude that (2.8) holds also for all nonnegative measurable functions defined on $M_\Lambda \times M_{\Lambda'}$ where the integrals in (2.8) may be equal to $+\infty$.

Lemma 2.9. Let $\Lambda \in \mathfrak{R}^d$ and g: $M \to [0, \infty)$ be a measurable function. Then

$$\int F(d\varphi)\, g(\varphi) = \int (F_\Lambda \times F_{\Lambda^c})(d[\varphi_1, \varphi_2])\, g(\varphi_1 + \varphi_2) \tag{2.10}$$

Proof. Setting $h(\varphi_1, \varphi_2) = g(\varphi_1 + \varphi_2)$, we get $h(\varphi_\Lambda, \varphi_{\Lambda^c}) = g(\varphi)$. So Lemma 2.9 is an immediate consequence of Lemma 2.7 and Remark 2.8.

Proof of Proposition 2.6. It is known (e.g., Ref. 11, Theorem II.10) that there exists a unique isomorphism $J_{\Lambda,\Lambda'}$ between $\mathscr{M}_\Lambda \otimes \mathscr{M}_{\Lambda'}$ and $L_2(M_\Lambda \times M_{\Lambda'}, F_\Lambda \times F_{\Lambda'})$ such that

$$J_{\Lambda,\Lambda'}(\Psi \otimes \Psi')(\varphi, \varphi') = \Psi(\varphi)\, \Psi'(\varphi')$$
$$\Psi \in \mathscr{M}_\Lambda, \quad \Psi' \in \mathscr{M}_{\Lambda'}, \quad \varphi \in M_\Lambda, \quad \varphi' \in M_{\Lambda'} \tag{2.11}$$

From (2.8) we get immediately that $U_{\Lambda,\Lambda'}$ given by

$$U_{\Lambda,\Lambda'} \Psi(\varphi) = \Psi(\varphi_\Lambda, \varphi_{\Lambda'}), \qquad \Psi \in L_2(M_\Lambda \times M_{\Lambda'}, F_\Lambda \times F_{\Lambda'}), \quad \varphi \in M_{\Lambda \cup \Lambda'} \tag{2.12}$$

defines a unitary operator from $L_2(M_\Lambda \times M_{\Lambda'}, F_\Lambda \times F_{\Lambda'})$ onto $\mathscr{M}_{\Lambda \cup \Lambda'}$. Putting $I_{\Lambda,\Lambda'} = U_{\Lambda,\Lambda'} J_{\Lambda,\Lambda'}$, we get the desired isomorphism with property (2.6). The uniqueness of such an isomorphism follows from the fact that all spaces $\mathscr{M}_\Lambda$, $\Lambda \in \mathfrak{R}^d$, are separable Hilbert spaces and that $(\Psi_k(v_\Lambda \cdot)\, \Psi'_n(v_{\Lambda'} \cdot))_{k,n \geqslant 0}$ is an orthonormal base in $\mathscr{M}_{\Lambda \cup \Lambda'}$ if $(\Psi_k)_{k \geqslant 0}$ and $(\Psi'_n)_{n \geqslant 0}$ are orthonormal bases in $\mathscr{M}_\Lambda$ and $\mathscr{M}_{\Lambda'}$ resp. (e.g., Ref. 11, Chapter II.4, Proposition 2). This ends the proof.

2.3. The Local Algebras

Denote by $\mathscr{A}_\Lambda = \mathscr{L}(\mathscr{M}_\Lambda)$ the von Neumann algebra of all bounded linear operators on $\mathscr{M}_\Lambda$, $\Lambda \in \mathfrak{R}^d$. For definition and properties of von Neumann algebras see, e.g., Ref. 2, Chapter 2, §1.5, and Ref. 1, Chapter 2.4.

For $\Lambda \in \mathfrak{B}$ we denote by ${}_\Lambda\mathscr{A}$ the natural imbedding of $\mathscr{A}_\Lambda$ into $\mathscr{L}(\mathscr{M})$. More precisely, for $\Lambda \in \mathfrak{B}$ we put ${}_\Lambda I = I_{\Lambda,\Lambda^c}$ [cf. (2.6)] and define a mapping ${}_\Lambda J$: $\mathscr{A}_\Lambda \to \mathscr{L}(\mathscr{M})$ by

$${}_\Lambda J A = {}_\Lambda I (A \otimes \mathbf{1}_{\Lambda^c})\, {}_\Lambda I^{-1}, \qquad A \in \mathscr{A}_\Lambda \tag{2.13}$$

where $\mathbf{1}_{\Lambda^c}$ is the identity operator in $\mathscr{A}_{\Lambda^c}$.

Definition 2.10. Ler $\Lambda \in \mathfrak{B}$. The subalgebra ${}_\Lambda\mathscr{A}$ of $\mathscr{L}(\mathscr{M})$ defined by ${}_\Lambda\mathscr{A} = \{{}_\Lambda JA\colon A \in \mathscr{A}_\Lambda\}$ is called the local algebra on Λ.

It is easy to check that ${}_\Lambda J$ is a $*$-isomorphism between $\mathscr{A}_\Lambda$ and ${}_\Lambda\mathscr{A}$ and that $\|{}_\Lambda JA\| = \|A\|_\Lambda$ for all $A \in \mathscr{A}_\Lambda$. We will call $\mathscr{A}_\Lambda$ and ${}_\Lambda\mathscr{A}$ briefly isomorphic (for a definition of $*$-isomorphisms see Ref. 1, Chapter 2.3.1). Working with ${}_\Lambda\mathscr{A}$ instead of $\mathscr{A}_\Lambda$ has the advantage that all ${}_\Lambda\mathscr{A}$, $\Lambda \in \mathfrak{B}$, have a common identity denoted in the sequel by $\mathbf{1}$, and that ${}_\Lambda\mathscr{A} \subseteq {}_{\Lambda'}\mathscr{A}$ for $\Lambda, \Lambda' \in \mathfrak{B}$, $\Lambda \subseteq \Lambda'$. Further, we have that A and A' commute for $A \in {}_\Lambda\mathscr{A}$, $A' \in {}_{\Lambda'}\mathscr{A}$, $\Lambda \cap \Lambda' = \varnothing$, $\Lambda, \Lambda' \in \mathfrak{B}$. Obviously, ${}_\Lambda\mathscr{A}$ is also a von Neumann algebra. All these facts are well known in the algebraic approach to quantum mechanics (e.g., Ruelle,[12,13] Emch,[2] Chapter 4, §1, Bratteli and Robinson,[1] Chapter 2.6).

We will show that the relation between ${}_\Lambda\mathfrak{M}$ and $\mathfrak{M}_\Lambda$ corresponds to the relation between ${}_\Lambda\mathscr{A}$ and $\mathscr{A}_\Lambda$.

For arbitrary $Y \in \mathfrak{M}$ denote by O_Y the operator of multiplication by the indicator function of Y, i.e.,

$$(O_Y \Psi)(\varphi) = \chi_Y(\varphi)\, \Psi(\varphi) \qquad \Psi \in \mathscr{M},\ \varphi \in M \tag{2.14}$$

Observe that for all $\Lambda \in \mathfrak{R}^d$ and $Y \in \mathfrak{M}_\Lambda$, $O_Y \in \mathscr{A}_\Lambda$. Since for all $\Lambda \in \mathfrak{R}^d$ and $Y \in \mathfrak{M}$, $v_\Lambda Y \in \mathfrak{M}^\Lambda$, we also have $O_{v_\Lambda Y} \in \mathscr{A}_\Lambda$ for all $Y \in \mathfrak{M}$.

Proposition 2.11. Let $\Lambda \in \mathfrak{B}$ and $Y \in \mathfrak{M}$. The following conditions are equivalent:

(i) $O_Y \in {}_\Lambda\mathscr{A}$.

(ii) There exists a $\tilde{Y} \in {}_\Lambda\mathfrak{M}$ such that $O_Y = O_{\tilde{Y}}$.

(iii) $O_Y = {}_\Lambda J O_{v_\Lambda Y}$.

Remark 2.12. From the proposition above we conclude that $\{{}_\Lambda J O_Y\colon Y \in \mathfrak{M}_\Lambda\} = \{O_Y\colon Y \in {}_\Lambda\mathfrak{M}\}$.

Proof of Proposition 2.11. We show the following chain of implications: (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii): Assume $Y \in {}_\Lambda\mathfrak{M}$. Immediately from the definition of ${}_\Lambda\mathfrak{M}$ it follows that $X \in {}_\Lambda\mathfrak{M}$ if and only if for all $\varphi \in M$, $\varphi_\Lambda \in v_\Lambda X$ implies $\varphi \in X$.

We thus get for all $\Psi \in \mathscr{M}$, $\varphi \in M$,

$$\begin{aligned}({}_\Lambda J O_{v_\Lambda Y} \Psi)(\varphi) &= {}_\Lambda I (O_{v_\Lambda Y} \otimes \mathbf{1}_{\Lambda^c})\, {}_\Lambda I^{-1} \Psi(\varphi) = \chi_{v_\Lambda Y}(\varphi_\Lambda)\, \Psi(\varphi)\\ &= \chi_Y(\varphi)\, \Psi(\varphi) = (O_Y \Psi)(\varphi)\end{aligned}$$

(iii) $\Rightarrow$ (i): From $O_{v_\Lambda Y} \in \mathscr{A}_\Lambda$ we conclude ${}_\Lambda J O_{v_\Lambda Y} \in {}_\Lambda\mathscr{A}$. From (iii) we thus get $O_Y \in {}_\Lambda\mathscr{A}$.

(i) ⇒ (ii): Observe that for all $Y \in \mathfrak{M}$ one has

$$\chi_Y(\varphi) = \chi_{v_A Y}(\varphi_A)\, \chi_{v_{A^c} Y}(\varphi_{A^c}), \qquad \varphi \in M \tag{2.15}$$

Consequently, for all $Y \in \mathfrak{M}$ we have

$$O_Y = {}_A I(O_{v_A Y} \otimes O_{v_{A^c} Y})\, {}_A I^{-1} \tag{2.16}$$

From the assumption $O_Y \in {}_A\mathscr{A}$ we thus get

$$O_{v_A Y} \otimes O_{v_{A^c} Y} = A \otimes \mathbf{1}_{A^c}$$

where A is an element of $\mathscr{A}_A$. Consequently,

$$A = O_{v_A Y} \qquad \text{and} \qquad O_{v_{A^c} Y} = \mathbf{1}_{A^c}$$

Observe that

$$O_{v_{A^c} Y} = \mathbf{1}_{A^c} \quad \text{implies} \quad O_{v_{A^c} Y} = O_{M_{A^c}}$$

Taking $\tilde{Y} \in \mathfrak{M}$ such that $v_A \tilde{Y} = v_A Y$ and $v_{A^c} \tilde{Y} = M_{A^c}$, we get $Y \in {}_A\mathfrak{M}$ and $O_{\tilde{Y}} = O_Y$. This ends the proof.

Now, for arbitrary $f \in \mathscr{M}$ denote by $\mathfrak{O}_f$ the operator of multiplication by the function f, i.e.,

$$(\mathfrak{O}_f \Psi)(\varphi) = f(\varphi)\, \Psi(\varphi), \qquad \Psi \in \mathscr{M},\ \varphi \in M \tag{2.17}$$

Observe that we have $O_Y = \mathfrak{O}_{\chi_Y}$ for all $Y \in \mathfrak{M}$.
We set

$$\mathscr{M}^b = \{f \in \mathscr{M} : f \text{ is essentially bounded with respect to } F\}$$

It is well known that $\{\mathfrak{O}_f : f \in \mathscr{M}^b\}$ is the maximal set of multiplication operators on $\mathscr{M}$ contained in $\mathscr{L}(\mathscr{M})$.

By the usual approximation procedure of measurable functions by step functions we get immediately the following result from Proposition 2.11:

Corollary 2.13. Let $f \in \mathscr{M}^b$, $A \in \mathfrak{B}$. The following conditions are equivalent:

(i) $\mathfrak{O}_f \in {}_A\mathscr{A}$.
(ii) There exists a ${}_A\mathfrak{M}$-measurable function $\tilde{f}$ such that $\mathfrak{O}_f = \mathfrak{O}_{\tilde{f}}$.
(iii) $\mathfrak{O}_f = {}_A J \mathfrak{O}_{f \cdot v_A}$.

2.4. The Quasilocal Algebra—Locally Normal States

Denote by $\mathscr{A}$ the uniform closure of $\bigcup_{\Lambda \in \mathfrak{B}} {}_\Lambda\mathscr{A}$, i.e., the closure with respect to the operator norm in $\mathscr{L}(\mathscr{M})$.

The pair $[\mathscr{A}, ({}_\Lambda\mathscr{A})_{\Lambda \in \mathfrak{B}}]$ is a quasilocal algebra (e.g., Ref. 2, Chapter 4, §1.1; Ref. 1, Definition 2.6.3).

Now, let $\tilde{\mathscr{A}}$ be a C^*-subalgebra of $\mathscr{L}(\mathscr{M})$. A positive normalized linear function η on $\tilde{\mathscr{A}}$ is called a *state* on $\tilde{\mathscr{A}}$.

A state η on a von Neumann subalgebra $\tilde{\mathscr{A}}$ of $\mathscr{L}(\mathscr{M})$ is called *normal* if there exists a density matrix ρ on $\mathscr{M}$ (i.e., a positive trace-class operator with trace one) such that

$$\eta(A) = \mathrm{Tr}(\rho A) \qquad (A \in \tilde{\mathscr{A}})$$

where Tr denotes the trace in $\mathscr{L}(\mathscr{M})$.

Definition 2.14. A state ω on $\mathscr{A}$ is said to be *locally normal* if for all $\Lambda \in \mathfrak{B}$ the restriction ${}_\Lambda\omega$ of ω to ${}_\Lambda\mathscr{A}$ is a normal state, i.e., for all $\Lambda \in \mathfrak{B}$ there exists a density matrix ${}_\Lambda\rho$ on $\mathscr{M}$ such that

$$\omega(A) = {}_\Lambda\omega(A) = \mathrm{Tr}({}_\Lambda\rho A), \qquad A \in {}_\Lambda\mathscr{A}$$

(e.g., Ref. 1, Definitions 2.6.6, 2.4.20, and Theorem 2.4.21).

Observe that a locally normal state ω on $\mathscr{A}$ is normal [more precisely, ω has a normal extension to $\mathscr{L}(\mathscr{M})$] if and only if there exists a density matrix ρ on $\mathscr{M}$ such that $\rho = {}_\Lambda\rho$ for all $\Lambda \in \mathfrak{B}$.

Remark 2.15. Without any difficulty we could consider more general quasilocal algebras $[\tilde{\mathscr{A}}, ({}_\Lambda\tilde{\mathscr{A}})_{\Lambda \in \mathfrak{B}}]$ with the property that for each $\Lambda \in \mathfrak{B}$, ${}_\Lambda\tilde{\mathscr{A}}$ is isomorphic to an irreducible C^*-subalgebra of $\mathscr{A}_\Lambda$. Since we are interested only in states locally determined by density matrices, we could extend such a state canonically to a locally normal state on $[\mathscr{A}, ({}_\Lambda\mathscr{A})_{\Lambda \in \mathfrak{B}}]$. For a more general definition of quasilocal algebras see the books cited above.

3. THE POSITION DISTRIBUTION OF A LOCALLY NORMAL STATE ON $\mathscr{A}$

In this section we want to relate to each locally normal state ω on $\mathscr{A}$ a point process Q_ω with the property that for all $\Lambda \in \mathfrak{B}$ and $Y \in {}_\Lambda\mathfrak{M}$, $Q_\omega(Y) = \omega(O_Y)$.

We will see that Q_ω is uniquely determined by the operators O_Y, $Y \in {}_\Lambda\mathfrak{M}$, $\Lambda \in \mathfrak{B}$.

The operators O_Y correspond to local position measurements and Q_ω may be interpreted as the position distribution of ω. Observe that the set $\{O_Y\colon Y \subseteq M \backslash M^f\}$ (i.e., operators O_Y, where Y consists only of infinite configurations) gives no contribution to the characterization of position measurements of the infinite system. Indeed, since $\Psi \in \mathscr{M}$ is concentrated on M^f, we have $O_Y \equiv 0$ for $Y \subseteq M \backslash M^f$. So the reasons for the need to describe the infinite point process Q_ω by the local position measurements are the same as for leaving the Fock space by passing over to infinite systems (Ref. 2, Chapter 1, §1).

Let ω be an arbitrary, but in the sequel fixed, locally normal state on $\mathscr{A}$. As above, we denote for each $\Lambda \in \mathfrak{B}$ by ${}_\Lambda\omega$ the restriction of ω to ${}_\Lambda\mathscr{A}$ and by ${}_\Lambda\rho$ the density matrix on $\mathscr{M}$ related to ${}_\Lambda\omega$. Since ${}_\Lambda\mathscr{A}$ and $\mathscr{A}_\Lambda$ are isomorphic, there exists a normal state ω_Λ on $\mathscr{A}_\Lambda$ and a density matrix ρ_Λ on $\mathscr{M}_\Lambda$ such that

$$\omega_\Lambda(A) = \mathrm{Tr}_\Lambda(\rho_\Lambda A(= \mathrm{Tr}[{}_\Lambda\rho({}_\Lambda JA)] = {}_\Lambda\omega({}_\Lambda JA) = \omega({}_\Lambda JA) \qquad A \in \mathscr{A}_\Lambda \tag{3.1}$$

[${}_\Lambda J$ was defined by (2.13)]. Here Tr_Λ denotes the trace in $\mathscr{A}_\Lambda$. For all $\Lambda \in \mathfrak{B}$ we define a point process Q_Λ by

$$Q_\Lambda(Y) \underset{\mathrm{def}}{=} \omega_\Lambda(O_{Y \cap M_\Lambda}), \qquad Y \in \mathfrak{M} \tag{3.2}$$

From (3.2) one easily gets for all $f \in \mathscr{M}_\Lambda \cap \mathscr{M}^b$

$$\mathbf{E}_{Q_\Lambda} f = \omega_\Lambda(\mathfrak{D}_f) = {}_\Lambda\omega({}_\Lambda J\mathfrak{D}_f) = \omega({}_\Lambda J\mathfrak{D}_f) \tag{3.3}$$

where

$$\mathbf{E}_{Q_\Lambda} f = \int Q_\Lambda(d\varphi)\, f(\varphi)$$

is the expectation of f with respect to the point process Q_Λ.

The Q_Λ is a point process of a special type. We have the following result.

Proposition 3.1. Let $\Lambda \in \mathfrak{B}$. The point process Q_Λ defined by (3.2) is a finite Σ^c-point process concentrated on M_Λ.

Proof. There exists an orthonormal system $(\Psi_j)_{j \in J}$ of elements from $\mathscr{M}_\Lambda$ and a sequence $(\alpha_j)_{j \in J}$, $\alpha_j \geqslant 0$, $\sum_{j \in J} \alpha_j = 1$, such that $\rho_\Lambda = \sum_{j \in J} \alpha_j(\Psi_j, \cdot)\, \Psi_j$. Having this representation of ρ_Λ, we get, for all $Y \in \mathscr{M}_\Lambda$,

$$\begin{aligned} Q_\Lambda(Y) &= \mathrm{Tr}_\Lambda(\rho_\Lambda O_Y) \\ &= \sum_{j\in J} \alpha_j(\Psi_j, O_Y \Psi_j) \\ &= \int_{M_\Lambda} F(d\varphi)\,\chi_Y(\varphi) \sum_{j\in J} \alpha_j |\Psi_j(\varphi)|^2 \\ &= \chi_Y(\mathfrak{o}) \sum_{j\in J} \alpha_j\, |\Psi_j(\mathfrak{o})|^2 \\ &= +\sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} d\underline{x}^n\, \chi_Y(\delta_{\underline{x}^n}) \sum_{j\in J} \alpha_j\, |\Psi_j(\delta_{\underline{x}^n})|^2 \end{aligned}$$

Wakolbinger and Eder (Ref. 14, Theorem 2.13) proved that for a finite point process Q to be Σ^c it is necessary and sufficient that there exist real functions λ_n on $\mathbb{R}^n$, $n \geqslant 1$, $\lambda_0 \in \mathbb{R}^1$, such that

$$Q(Y) = \chi_Y(\mathfrak{o})\,\lambda_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \int d\bar{x}^n\, \chi_Y(\delta_{\underline{x}^n})\, \lambda_n(\underline{x}_n)$$

Consequently, putting

$$\lambda_n(\underline{x}^n) = \chi_{\Lambda^n}(\underline{x}^n) \sum_{j\in J} \alpha_j\, |\Psi_j(\delta_{\underline{x}^n})|^2, \qquad n \geqslant 1$$

$$\lambda_0 = \sum_{j\in J} \alpha_j\, |\Psi_j(\mathfrak{o})|^2$$

we get that Q_Λ is a Σ^c-point process.

Now, we define for each $\Lambda \in \mathfrak{B}$ a probability measure ${}_\Lambda Q$ on $[M, {}_\Lambda\mathfrak{M}]$ by

$$_\Lambda Q(Y) = Q_\Lambda(v_\Lambda Y), \qquad Y \in {}_\Lambda\mathfrak{M} \tag{3.4}$$

Since ${}_\Lambda\mathfrak{M} = \{v_\Lambda^{-1} Y\colon Y \in \mathfrak{M}_\Lambda\}$, the probability measure ${}_\Lambda Q$ is well-defined. From (3.1) and (3.2) we get

$$_\Lambda Q(Y) = \omega_\Lambda(O_{v_\Lambda Y}) = {}_\Lambda\omega({}_\Lambda J O_{v_\Lambda Y}) = {}_\Lambda\omega(O_Y) = \omega(O_Y) \qquad Y \in {}_\Lambda\mathfrak{M} \tag{3.5}$$

Now, for all $\Lambda, \Lambda' \in \mathfrak{B}$ such that $\Lambda \subseteq \Lambda'$ we have ${}_\Lambda\mathfrak{M} \subseteq {}_{\Lambda'}\mathfrak{M}$, and because of the quasilocal structure, ${}_\Lambda\mathscr{A} \subseteq {}_{\Lambda'}\mathscr{A}$. So, for each $Y \in {}_\Lambda\mathfrak{M}$ we have $O_Y \in {}_\Lambda\mathscr{A}$, $O_Y \in {}_{\Lambda'}\mathscr{A}$. From ${}_\Lambda\omega(O_Y) = {}_{\Lambda'}\omega(O_Y)$ $[=\omega(O_Y)]$ we conclude that

$$_\Lambda Q(Y) = {}_{\Lambda'}Q(Y), \qquad \Lambda, \Lambda' \in \mathfrak{B}, \quad \Lambda \subseteq \Lambda', \quad Y \in {}_\Lambda\mathfrak{M} \tag{3.6}$$

This allows us to prove the following result:

Theorem 3.2. There exists exactly one point process Q_ω such that

$$Q_\omega(Y) = \omega(O_Y), \qquad \Lambda \in \mathfrak{B}, \quad Y \in {}_\Lambda\mathfrak{M} \tag{3.7}$$

Proof. Let n be a natural number, $\Lambda_j \in \mathfrak{B}$, $k_j \in \mathbf{N}$, for $j \in \{1,..., n\}$, $\Lambda_i \cap \Lambda_j = \emptyset$ for $i \neq j$. There exists a set $\Lambda \in \mathfrak{B}$ such that $\Lambda \supseteq \Lambda_1 \cup \cdots \cup \Lambda_n$. We set

$$p^{\Lambda_1,...,\Lambda_n}([k_1,..., k_n]) = {}_\Lambda Q(\{\varphi \in M: \varphi(\Lambda_1) = k_1,..., \varphi(\Lambda_n) = k_n\}) \tag{3.8}$$

It is easy to check that $p^{\Lambda_1,...,\Lambda_n}$ defines a probability distribution on $\mathbf{N}^n$. Since

$$\{\varphi \in M: \varphi(\Lambda_1) = k_1,..., \varphi(\Lambda_n) = k_n\} \in {}_\Lambda\mathfrak{M}$$

from (3.6) we conclude that $p^{\Lambda_1,...,\Lambda_n}$ does not depend on the special choice of the set $\Lambda \supseteq \Lambda_1 \cup \cdots \Lambda_n$. It is easy to check that the probability distributions $p^{\Lambda_1,...,\Lambda_n}$ on $\mathbf{N}^n$ satisfy the following compatibility conditions:

(i) For all permutations $(i_1,..., i_n)$ of $(1,..., n)$ and for all $k_1,..., k_n \in \mathbf{N}$ the following holds:

$$p^{\Lambda_1,...,\Lambda_n}([k_1,..., k_n]) = p^{\Lambda_{i_1},...,\Lambda_{i_n}}([k_{i_1},..., k_{i_n}])$$

(ii) For all $k_1,..., k_{n-1} \in \mathbf{N}$ we have

$$\sum_{k \in \mathbf{N}} p^{\Lambda_1,...,\Lambda_n}([k_1,..., k_{n-1}, k]) = p^{\Lambda_1,...,\Lambda_{n-1}}([k_1,..., k_{n-1}])$$

(iii) For all $m \in \{1,..., n\}$ and for all $k_1,..., k_m \in \mathbf{N}$ the following holds:

$$\begin{aligned} &p^{\Lambda_1,...,\Lambda_n}(\{[l_1,..., l_n]: l_1 = k_1,..., l_{m-1} = k_{m-1}, l_m + \cdots + l_n = k_m\}) \\ &\quad = p^{\Lambda_1,...,\Lambda_{m-1}, \Lambda_m \cup \cdots \cup \Lambda_n}([k_1,..., k_m]) \end{aligned}$$

(v) For all sequences $(\Lambda_{n,1} \cup \cdots \cup \Lambda_{n,m_n})_{n \geqslant 1}$ of finite unions of pairwise disjoint sets from $\mathfrak{B}$ decreasing monotonically toward $\emptyset$ we have

$$\lim_{n \to \infty} p^{\Lambda_{n,1},...,\Lambda_{n,m_n}}([0,..., 0]) = 1$$

In Ref. 10, Theorem 1.3.5, it is shown (in analogy to the classical Kolmogorov theorem on finite-dimensional distributions) that the conditions (i)–(iv) are sufficient (and necessary) for the existence of exactly one point process Q_ω with the property

$$\begin{aligned} &Q_\omega(\{\varphi \in M: \varphi(\Lambda_1) = k_1,..., \varphi(\Lambda_n) = k_n\}) \\ &\quad = p^{\Lambda_1,...,\Lambda_n}([k_1,..., k_n]) \end{aligned}$$

for all $n \geqslant 1$, $k_1,..., k_n \in \mathbf{N}$, $\Lambda_1,..., \Lambda_n \in \mathfrak{B}$, $\Lambda_j \cap \Lambda_k = \emptyset$ for $j \neq k$.

We thus finally get that there exists a unique point process Q_ω such that $Q_\omega(Y) = {}_\Lambda Q(Y)$ for all Y of the form

$$\{\varphi \in M\colon \varphi(\Lambda_1) = k_1, \ldots, \varphi(\Lambda_n) = k_n\}$$

where $\Lambda \supseteq \Lambda_1 \cup \cdots \cup \Lambda_n$, Λ, $\Lambda_j \in \mathfrak{B}$. Since for each $\Lambda \in \mathfrak{B}$, ${}_\Lambda\mathfrak{M}$ is generated by sets of this type, we get (3.7).

Each position measurement can be expressed by a multiplication operator $\mathfrak{O}_g$, where g is a (real-valued) measurable function on M. Now, a position measurement in $\bigcup_{\Lambda \in \mathfrak{B}} {}_\Lambda\mathscr{A}$ corresponds to a position measurement in some ${}_\Lambda\mathscr{A}$ and thus to a multiplication operator $\mathfrak{O}_g \in {}_\Lambda\mathscr{A}$. Because of Corollary 2.13, g is a (real-valued) ${}_\Lambda\mathfrak{M}$-measurable function from $\mathscr{M}^b$. From (3.7), (3.4), and (3.3) one easily concludes that for all $\Lambda \in \mathfrak{B}$ and $\mathfrak{O}_g \in {}_\Lambda\mathscr{A}$

$$\omega(\mathfrak{O}_g) = \mathbf{E}_{{}_\Lambda Q}\, g = \mathbf{E}_{Q_\omega}\, g$$

Thus, for each position measurement $\mathfrak{O}_g$ the expectation $\omega(\mathfrak{O}_g)$ is equal to the expectation of g with respect to the point process Q_ω. On the other hand, Q_ω is uniquely determined by all of the integrals $\mathbf{E}_{Q_\omega} g$, $\mathfrak{O}_g \in \bigcup_{\Lambda \in \mathfrak{B}} {}_\Lambda\mathscr{A}$ (g real-valued). This allows an interpretation of Q_ω as the distribution of outcomes of position measurements of a quantum system being in the state ω. That is why we will call Q_ω the *position distribution of* ω.

From (3.7), (3.5), and (3.4) we conclude that for all $\Lambda \in \mathfrak{B}$ and $Y \in {}_\Lambda\mathfrak{M}$

$$Q_\omega(Y) = {}_\Lambda Q(Y) = Q_\Lambda(v_\Lambda Y)$$

Thus, from Proposition 3.1 we have that the position distribution Q_ω is locally a finite Σ^c-point process concentrated on $[M_\Lambda, \mathfrak{M}_\Lambda]$. From Theorems 2.11 and 2.13 in Ref. 14 we know that each Σ^c-point process is locally a finite Σ^c-point process.

On the other hand, we have the following result:

Theorem 3.3. Let Q be a (possibly infinite) Σ^c-point process. There exists (at least one) locally normal state ω on $\mathscr{A}$ with position distribution $Q_\omega = Q$, i.e., such that

$$\omega(O_Y) = Q(Y), \qquad Y \in {}_\Lambda\mathfrak{M},\ \Lambda \in \mathfrak{B}$$

Before we prove this theorem, we will introduce an important notion from point process theory.

Definition 3.4. Let Q be a point process. We define a measure $C_Q^{(\infty)}$ on $[M \times M, \mathfrak{M} \otimes \mathfrak{M}]$ characterized by

$$C_Q^{(\infty)}(Y_1 \times Y_2) = \int Q(d\varphi) \sum_{\substack{\hat{\varphi} \subseteq \varphi \\ \hat{\varphi} \in M^f}} \chi_{Y_1}(\hat{\varphi})\, \chi_{Y_2}(\varphi - \hat{\varphi}), \qquad Y_1, Y_2 \in \mathfrak{M} \tag{3.9}$$

Here $\hat{\varphi} \subseteq \varphi$ means that $\varphi - \hat{\varphi} \in M$, i.e., $\hat{\varphi}$ is a subconfiguration of φ. The measure $C_Q^{(\infty)}$ is called the compound Campbell measure of Q (cf. Ref. 8, Chapter 12.3). We collect some properties of $C_Q^{(\infty)}$:

Lemma 3.5. Let Q be a point process. Then:

(i) $C_Q^{(\infty)}$ is σ-finite.

(ii) $C_Q^{(\infty)}$ is concentrated on $M^f \times M$.

(iii) $C_Q^{(\infty)}(M_\Lambda \times M_{\Lambda'}) = Q(M_{\Lambda \cup \Lambda'})$ for all $\Lambda \in \mathrm{B}$, $\Lambda' \in \mathfrak{R}^d$, $\Lambda \cap \Lambda' = \varnothing$.

(iv) $C_Q^{(\infty)}(\{\mathfrak{o}\} \times (\cdot)) = Q$.

(v) If Q is a Σ^c-point process, then there exists a Σ^c-point process P such that $C_Q^{(\infty)} \ll F \times P$.

Proof. For a proof of (i)–(iv) see Ref. 8, Chapter 12.3. We will show (v). Let Q be a Σ^c-process. Then for all $n \geqslant 1$ there exists a Σ^c-process P_n such that $C_Q^{(n)} \ll l^n \times P_n$ (where l^n denotes the Lebesgue measure on $\mathbb{R}^{nd}$) (Ref. 14, Theorems 2.9 and 2.10). Let P be a Σ^c-process dominating all P_n, $n \in \mathbf{N}$ $(P_0 = Q)$ (for instance take $P = \sum_{n \in \mathbf{N}} \alpha_n P_n$, where $\alpha_n > 0$ for all $n \in \mathbf{N}$, $\sum_{n \in \mathbf{N}} \alpha_n = 1$). We thus get for all $n \geqslant 1$, $C_Q^{(n)} \ll l^n \times P$. Setting $C_Q^{(0)} = Q$ and using the notation (2.7), we obtain for all Y_1, $Y_2 \in \mathfrak{M}$

$$\begin{aligned}
C_Q^{(\infty)}(Y_1 \times Y_2) &= \sum_{n \in \mathbf{N}} C_Q^{(\infty)}[(Y_1 \cap M_n) \times Y_2] \\
&= \sum_{n \in \mathbf{N}} \frac{1}{n!} \int C_Q^{(n)}(d[\underline{x}^n, \varphi])\, \chi_{Y_1}(\delta_{\underline{x}^n})\, \chi_{Y_2}(\varphi) \\
&= \int_{Y_2} P(d\varphi) \sum_{n \in \mathbf{N}} \frac{1}{n!} \int d\underline{x}^n\, \frac{dC_Q^{(n)}}{d(l^n \times P)}(\underline{x}^n, \varphi)\, \chi_{Y_1}(\delta_{\underline{x}^n}) \\
&= \int_{Y_2} P(d\varphi) \int_{Y_1} F(d\hat{\varphi})\, \kappa_Q^P(\hat{\varphi}, \varphi)
\end{aligned} \tag{3.10}$$

where we have set

$$\kappa_Q^P(\delta_{\underline{x}^n}, \varphi) = \frac{dC_Q^{(n)}}{d(l^n \times P)}(\underline{x}^n, \varphi), \qquad n \geqslant 1, \quad \underline{x}^n \in \mathbb{R}^{nd}, \quad \varphi \in M \tag{3.11}$$

and

$$\kappa_Q^P(\mathfrak{o}, \varphi) = \frac{dQ}{dP}(\varphi), \qquad \varphi \in M \tag{3.12}$$

This ends the proof.

Lemma 3.6. Let Y be a Σ^c-point process, and let P and R be Σ^c-point processes such that

$$C_Q^{(\infty)} \ll F \times P, \qquad C_P^{(\infty)} \ll F \times R \tag{3.13}$$

Then, denoting by κ_Q^P and κ_P^R (versions of) the Radon–Nikodym derivatives $dC_Q^{(\infty)}/d(F \times P)$ and $dC_P^{(\infty)}/d(F \times R)$, resp., we have for all $\Lambda, \Lambda' \in \mathfrak{B}$, $\Lambda \cap \Lambda' = \varnothing$

$$\kappa_Q^P(\varphi_1 + \varphi_2, \varphi_3)\, \kappa_P^R(\mathfrak{o}, \varphi_3) = \kappa_Q^P(\varphi_1, \varphi_2 + \varphi_3)\, \kappa_P^R(\varphi_2, \varphi_3) \tag{3.14}$$

$F \times F \times R$-a.a. $(\varphi_1, \varphi_2, \varphi_3)$ from $M_\Lambda \times M_{\Lambda'} \times M_{(\Lambda \cup \Lambda')^c}$. [The existence of P and R with (3.13) follows from Lemma 3.5, (v).]

Proof. We fix $\Lambda, \Lambda' \in \mathfrak{B}$, $\Lambda \cap \Lambda' = \varnothing$, and set $\Lambda'' = (\Lambda \cup \Lambda')^c$. For each measurable function f: $M_\Lambda \times M_{\Lambda'} \times M_{\Lambda''} \to [0, \infty)$ we get, using Lemma 2.7 and Lemma 3.5, part (iii),

$$\begin{aligned}
&\int_{M_\Lambda \times M_{\Lambda'} \times M_{\Lambda''}} (F \times F \times R)(d[\varphi_1, \varphi_2, \varphi_3]) \\
&\qquad \times \kappa_Q^P(\varphi_1 + \varphi_2, \varphi_3)\, \kappa_P^R(\mathfrak{o}, \varphi_3)\, f(\varphi_1, \varphi_2, \varphi_3) \\
&\quad = \int_{M_{\Lambda \cup \Lambda'}} F(d\varphi_1) \int_{M_{\Lambda''}} R(d\varphi_2)\, \kappa_P^R(\mathfrak{o}, \varphi_2)\, \kappa_Q^P(\varphi_1, \varphi_2)\, f(\varphi_{1_\Lambda}, \varphi_{1_{\Lambda'}}, \varphi_2) \\
&\quad = \int_{M_{\Lambda \cup \Lambda'}} F(d\varphi_1) \int_{M_{\Lambda''}} P(d\varphi_2)\, \kappa_P^Q(\varphi_1, \varphi_2)\, f(\varphi_{1_\Lambda}, \varphi_{1_{\Lambda'}}, \varphi_2) \\
&\quad = \int C_Q^{(\infty)}(d[\varphi_1, \varphi_2])\, \chi_{M_{\Lambda \cup \Lambda'}}(\varphi_1)\, \chi_{M_{\Lambda''}}(\varphi_2)\, f(\varphi_{1_\Lambda}, \varphi_{1_{\Lambda'}}, \varphi_2) \\
&\quad = \int Q(d\varphi)\, f(\varphi_\Lambda, \varphi_{\Lambda'}, \varphi_{\Lambda''}) \\
&\quad = \int Q(\varphi) \sum_{\varphi_1 \subseteq \varphi} \chi_{M_\Lambda}(\varphi_1)\, \chi_{M_{\Lambda^c}}(\varphi - \varphi_1)\, f(\varphi_1, (\varphi - \varphi_1)_{\Lambda'}, (\varphi - \varphi_1)_{\Lambda''}) \\
&\quad = \int P(d\varphi) \int_{M_\Lambda} F(d\varphi_1)\, \kappa_Q^P(\varphi_1, \varphi)\, \chi_{M_{\Lambda^c}}(\varphi)\, f(\varphi_1, \varphi_{\Lambda'}, \varphi_{\Lambda''})
\end{aligned}$$

$$= \int_{M_\Lambda} F(d\varphi_1) \int P(d\varphi) \sum_{\varphi_2 \subseteq \varphi} \chi_{M_{\Lambda'}}(\varphi_2)\, \chi_{M_{\Lambda''}}(\varphi - \varphi_2)$$

$$\times \kappa_Q^P(\varphi_1, \varphi)\, f(\varphi_1, \varphi_2, \varphi - \varphi_2)$$

$$= \int_{M_\Lambda} F(d\varphi_1) \int_{M_{\Lambda'}} F(d\varphi_2) \int_{M_{\Lambda''}} R(d\varphi_3)\, \kappa_P^R(\varphi_2, \varphi_3)$$

$$\times \kappa_Q^P(\varphi_1, \varphi_2 + \varphi_3)\, f(\varphi_1, \varphi_2, \varphi_3)$$

This proves (3.14).

Proof of Theorem 3.3. Let Q be a Σ^c-point process, and let P and R be Σ^c-processes with (3.13). Again we denote by κ_Q^P and κ_P^R (versions of) the Radon–Nikodym derivative $dC_Q^{(\infty)}/d(F \times P)$, resp. $dC_P^{(\infty)}/d(F \times R)$. Without loss of generality we may assume $\kappa_Q^P(\varphi_1, \varphi_2) \geqslant 0$ for all $\varphi_1, \varphi_2 \in M$. We put

$$\Psi(\varphi_1, \varphi_2) = [\kappa_Q^P(\varphi_1, \varphi_2)]^{1/2}, \qquad \varphi_1, \varphi_2 \in M \tag{3.15}$$

and for all $\Lambda \in \mathfrak{B}$

$$\Psi_\varphi^\Lambda(\cdot) = \Psi(\cdot, \varphi)\, \chi_{M_\Lambda}(\cdot)\, \chi_{M_{\Lambda^c}}(\varphi), \qquad \varphi \in M \tag{3.16}$$

First we will prove that for all $\Lambda \in \mathfrak{B}$ and P-a.a. φ (and thus for Q-a.a. φ) we have $\Psi_\varphi^\Lambda \in \mathscr{M}_\Lambda$. Indeed, from Lemma 3.5, part (iii), we get

$$\int_M P(d\varphi) \int_M F(d\hat{\varphi})\, |\Psi_\varphi^\Lambda(\hat{\varphi})|^2$$

$$= \int_{M_{\Lambda^c}} P(d\varphi) \int_{M_\Lambda} F(d\hat{\varphi})\, \kappa_Q^P(\hat{\varphi}, \varphi)$$

$$= C_Q^{(\infty)}(M_\Lambda \times M_{\Lambda^c}) = Q(M) = 1 \tag{3.17}$$

Consequently, for all $\Lambda \in \mathfrak{B}$ and P-a.a. φ, $\|\Psi_\varphi^\Lambda\| < \infty$. Thus, for all $\Lambda \in \mathfrak{B}$, by

$${}_\Lambda\omega(A) = \int_{M_{\Lambda^c}} P(d\varphi)(\Psi_\varphi^\Lambda, O_{M_\Lambda} A O_{M_\Lambda} \Psi_\varphi^\Lambda), \qquad A \in {}_\Lambda\mathscr{A} \tag{3.18}$$

there is defined a positive linear functional on ${}_\Lambda\mathscr{A}$. From (3.17) we further conclude

$${}_\Lambda\omega(\mathbf{1}) = \int_{M_{\Lambda^c}} P(d\varphi)(\Psi_\varphi^\Lambda, \Psi_\varphi^\Lambda) = C_Q^{(\infty)}(M_\Lambda \times M_{\Lambda^c}) = 1$$

Consequently, for each $\Lambda \in \mathfrak{B}$ there is defined a state on ${}_\Lambda\mathscr{A}$.

We will prove that all states ${}_\Lambda\omega$ are normal ones. We fix $\Lambda \in \mathfrak{B}$. There exists a sequence of (finite rank) projectors $(A_n)_{n\geq 1}$ in $\mathscr{A}_\Lambda$ such that $(0\leq)\, A_n \leq A_{n+1}$ for all $n \geq 1$ and A_n converges σ-weakly to $\mathbf{1}_\Lambda$.

This implies

$$\lim_{n\to\infty} (\Psi_\varphi^\Lambda, A_n \Psi_\varphi^\Lambda) = (\Psi_\varphi^\Lambda, \Psi_\varphi^\Lambda), \qquad P\text{-a.a. } \varphi \tag{3.19}$$

By Lebesgue's dominated convergence theorem, we get from (3.19)

$$\lim_{n\to\infty} {}_\Lambda\omega({}_\Lambda J A_n) = \int_{M_{\Lambda^c}} P(d\varphi)(\Psi_\varphi^\Lambda, \Psi_\varphi^\Lambda) = 1$$

Because of the isomorphism between $\mathscr{A}_\Lambda$ and ${}_\Lambda\mathscr{A}$, we obtain, using Theorem 2.6.14 in Ref. 1, that ${}_\Lambda\omega$ is a normal state on ${}_\Lambda\mathscr{A}$.

We still have to prove compatibility of all states ${}_\Lambda\omega$, i.e., we have to show

$${}_\Lambda\omega(A) = {}_{\Lambda'}\omega(A), \qquad \Lambda, \Lambda' \in \mathfrak{B}, \quad \Lambda \subseteq \Lambda', \quad A \in {}_\Lambda\mathscr{A} \tag{3.20}$$

We fix $\Lambda, \Lambda' \in \mathfrak{B}$, $\Lambda \subseteq \Lambda'$, and $A \in {}_\Lambda\mathscr{A}$.

Using Lemma 2.7, Lemma 3.5, part (iii), Lemma 3.6, and the fact that $O_{M_{\Lambda'}} A O_{M_{\Lambda'}}$ is isomorphic to $O_{M_\Lambda} A O_{M_\Lambda} \otimes \mathbf{1}_{\Lambda'\setminus\Lambda}$, we get the following chain of equalities:

$$\begin{aligned}
{}_{\Lambda'}\omega(A) &= \int_{M_{\Lambda'^c}} P(d\varphi) \int_{M_{\Lambda'}} F(d\hat\varphi)[\kappa_Q^P(\hat\varphi, \varphi)]^{1/2}\, O_{M_{\Lambda'}} A O_{M_{\Lambda'}} [\kappa_Q^P(\hat\varphi, \varphi)]^{1/2} \\
&= \int_{M_{\Lambda'^c}} R(d\varphi_3)\, \kappa_P^R(\mathfrak{o}, \varphi_3) \int_{M_{\Lambda'\setminus\Lambda}} F(d\varphi_2) \int_{M_\Lambda} F(d\varphi_1) \\
&\quad \times [\kappa_Q^P(\varphi_1 + \varphi_2, \varphi_3)]^{1/2}\, O_{M_{\Lambda'}} A O_{M_{\Lambda'}} [\kappa_Q^P(\varphi_1 + \varphi_2, \varphi_3)]^{1/2} \\
&= \int_{M_{\Lambda'^c}} R(d\varphi_3) \int_{M_{\Lambda'\setminus\Lambda}} F(d\varphi_2) \int_{M_\Lambda} F(d\varphi_1) \\
&\quad \times [\kappa_P^R(\mathfrak{o}, \varphi_3)\, \kappa_Q^P(\varphi_1 + \varphi_2, \varphi_3)]^{1/2} \\
&\quad \times O_{M_{\Lambda'}} A O_{M_{\Lambda'}} [\kappa_P^R(\mathfrak{o}, \varphi_3)]^{1/2} [\kappa_Q^P(\varphi_1 + \varphi_2, \varphi_3)]^{1/2} \\
&= \int_{M_{\Lambda'^c}} R(d\varphi_3) \int_{M_{\Lambda'\setminus\Lambda}} F(d\varphi_2) \int_{M_\Lambda} F(d\varphi_1)\, \kappa_P^R(\varphi_2, \varphi_3) \\
&\quad \times [\kappa_Q^P(\varphi_1, \varphi_3 + \varphi_2)]^{1/2}\, O_{M_{\Lambda'}} A O_{M_{\Lambda'}} [\kappa_P^R(\varphi_2, \varphi_3)]^{1/2} \\
&\quad \times [\kappa_Q^P(\varphi_1, \varphi_3 + \varphi_2(]^{1/2}
\end{aligned}$$

$$= \int_{M_{\Lambda'^c}} R(d\varphi_3) \int_{M_{\Lambda'\setminus\Lambda}} F(d\varphi_2)\, \kappa_P^R(\varphi_2, \varphi_3) \int_{M_\Lambda} F(d\varphi_1)$$

$$\times [\kappa_Q^P(\varphi_1, \varphi_3 + \varphi_2)]^{1/2}\, O_{M_{\Lambda'}} A O_{M_{\Lambda'}} [\kappa_Q^P(\varphi_1, \varphi_3 + \varphi_2)]^{1/2}$$

$$= \int_{M_{\Lambda'^c}} R(d\varphi_3) \int_{M_{\Lambda'\setminus\Lambda}} F(d\varphi_2)\, \kappa_P^R(\varphi_2, \varphi_3) \int_{M_\Lambda} F(d\varphi_1)$$

$$\times \Psi^\Lambda_{\varphi_3 + \varphi_2}(\varphi_1)(O_{M_\Lambda} A O_{M_\Lambda} \Psi^\Lambda_{\varphi_3 + \varphi_2})(\varphi_1)$$

$$= \int_{M_{\Lambda^c}} P(d\varphi_3) \int_{M_\Lambda} F(d\varphi_1)\, \Psi^\Lambda_{\varphi_3}(\varphi_1)(O_{M_\Lambda} A O_{M_\Lambda} \Psi^\Lambda_{\varphi_3})(\varphi_1)$$

$$= \int_{M_{\Lambda^c}} P(d\varphi)(\Psi^\Lambda_\varphi, O_{M_\Lambda} A O_{M_\Lambda} \Psi^\Lambda_\varphi) = {}_\Lambda\omega(A) \tag{3.21}$$

This proves (3.20). We thus have that there exists a locally normal state ω on $\mathscr{A}$ such that

$$\omega(A) = {}_\Lambda\omega(A), \qquad \Lambda \in \mathfrak{B}, \quad A \in {}_\Lambda\mathscr{A}$$

Finally, using the identity

$$O_{M_\Lambda} O_Y O_{M_\Lambda} = O_{v_\Lambda Y}, \qquad \Lambda \in \mathfrak{B}, \quad Y \in {}_\Lambda\mathfrak{M}$$

we get for all $\Lambda \in \mathfrak{B}$ and $Y \in {}_\Lambda\mathfrak{M}$

$$\omega(O_Y) = {}_\Lambda\omega(O_Y)$$

$$= \int_{M_{\Lambda^c}} P(d\varphi)(\Psi^\Lambda_\varphi, 0_{v_\Lambda Y} \Psi^\Lambda_\varphi)$$

$$= \int_{M_{\Lambda^c}} P(d\varphi) \int_{v_\Lambda Y} F(d\hat\varphi)\, \kappa_Q^P(\hat\varphi, \varphi)$$

$$= \int Q(d\varphi) \sum_{\substack{\hat\varphi \subseteq \varphi \\ \hat\varphi \in M_f}} \chi_{v_\Lambda Y}(\hat\varphi)\, \chi_{M_{\Lambda^c}}(\varphi - \hat\varphi)$$

$$= \int Q(d\varphi)\, \chi_{v_\Lambda Y}(\varphi_\Lambda)\, \chi_{M_{\Lambda^c}}(\varphi_{\Lambda^c})$$

$$= \int Q(d\varphi)\, \chi_{v_\Lambda Y}(\varphi_\Lambda)$$

$$= \int Q(d\varphi)\, \chi_Y(\varphi) = Q(Y)$$

This shows that $Q = Q_\omega$ and ends the proof of Theorem 3.3.

Summarizing the results, we can state that to each locally normal state ω on $\mathscr{A}$ there corresponds a uniquely determined point process Q_ω—the position distribution of ω (Theorem 3.2). This point process is locally a Σ^c-process. It is still an open problem whether there exist (physically meaningful) locally normal states for which the position distributions are not Σ^c-processes.

If the locally normal state ω is a normal one [more precisely, if it has a normal extension to $\mathscr{L}(\mathscr{M})$], then its position distribution Q_ω is a finite Σ^c-process. On the other hand, to each finite Σ^c-process Q we can relate a normal state ω such that $Q_\omega = Q$.

Many examples are given in Ref. 5. For instance, one can define infinite Glauber states and prove that the position distributions of such states are Poisson point processes (cf. Ref. 5, Chapter 7).

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